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# SEMILINEAR EQUATIONS WITH DISSIPATIVE TIME-DEPENDENT DOMAIN PERTURBATIONS

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ABSTRACT

Let $X$ be a real Banach space and let $A : D(A) \subset X \to X$ be the (linear) infinitesimal generator of the semigroup $S(t)$ of class $C_0$ (of type $\omega$). Assume that the function $(t, x) \to F(t, x)$ is continuous, the domain $D(t) = D(F(t, \cdot))$ is such that $t \to D(t)$ is closed and for each $t \in (a, b)$, the operator $x \to F(t, x)$ is dissipative. One proves that the subtangential condition (A5) is necessary and sufficient for the existence of the mild solution to the equation $u' = Au + F(t, u)$. All previous results of this type are included here. An elementary method for proving the uniqueness is pointed out and applications to PDE are given.

## 1. Introduction. Statement of the main results

Let $X$ be a real Banach space of norm $\|\cdot\|$. Recall that a family $S = \{S(t); t \geqq 0\}$ of bounded linear operators $S(t) : X \to X$ is said to be a semigroup of class $C_0$ if

$$S(0) = I \text{ (the identity)}, \quad S(t+s) = S(t)S(s), \quad t, s \geqq 0, \tag{1.1}$$

$$\lim_{h \downarrow 0} S(h)x = x, \qquad \forall x \in X. \tag{1.2}$$

$S$ is said to be of type $\omega \in R$ if

$$\|S(t)\|_{L(X)} \leqq \exp(\omega t), \qquad 0 \leqq t < \infty. \tag{1.3}$$

$A : D(A) \subset X \to X$ is said to be the infinitesimal generator of $S$ if

$$Ax = \lim_{h \downarrow 0} (S(h)x - x)/h, \qquad \forall x \in D(A).$$

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Let us consider the initial value problem

$$u' = Au + F(t,x), \quad u(t_0) = x \in D(t_0), \quad t \in [t_0, b), \tag{1.4}$$

where $t_0 \in (a,b)$, $-\infty \leqq a < b \leqq \infty$, $D(t) = D(F(t,\cdot)) \subset X$.

If $J_u$ is a subinterval of $[t_0, b)$ with $t_0 \in J_u$, then $u$ is said to be a mild solution to (1.4) on $J_u$, if $u(t) \subset D(t)$ for all $t \in J_u$, $u$ is continuous on $J_u$ and satisfies the integral equation

$$u(t) = S(t-t_0)x + \int_{t_0}^{t} S(t-s)F(s,u(s))ds, \qquad t \in J_u. \tag{1.5}$$

If $y \in X$ and $D \subset X$ then $d(y;D)$ stands for the distance from $y$ to $D$. Recall that

$$|d(y;D) - d(z;D)| \leqq \|y - z\|, \qquad y, z \in X. \tag{1.6}$$

For the convenience of future reference we record the following conditions:

(A1) $A : D(A) \subset X \to X$ is the infinitesimal generator of the linear semigroup $S$ of class $C_0$ and of type $\omega \in R$.

(A2) For each $t_0 \in (a,b)$ and $x \in D(t_0)$, there exist $r > 0$ and $T \in (t_0, b)$ such that $D(t) \cap B(x,r)$ is nonempty for all $t \in [t_0, T]$ and $t \to D(t) \cap B(x,r)$ is closed on $[t_0, T]$. ($B(x,r) = \{y \in X; \|y - x\| \leqq r\}$.)

(A3) $(t,x) \to F(t,x)$ is continuous at each $(t,x)$ with $t \in (a,b)$, $x \in D(t)$.

(A4) For each $t \in (a,b)$ the operator $x \to F(t,x)$ is $g(t)$-dissipative, i.e.

$$(1 - \lambda g(t))\|x_1 - x_2\| \leqq \|x_1 - x_2 - \lambda(F(t,x_1) - F(t,x_2))\|$$

for all $x_1, x_2 \in D(t)$ and $\lambda > 0$, with $g : (a,b) \to [0, +\infty)$ a nondecreasing function.

(A5) $\liminf_{h \downarrow 0} h^{-1} d(S(h)x + hF(t,x); D(t+h)) = 0, \quad \forall t \in (a,b), \quad x \in D(t).$

REMARK 1.1. The mapping $t \to D(t)$ is said to be closed on $(a,b)$ if the conditions: $x_n \in D(t_n)$, $t_n \in (a,b)$, $t_n \to t \in (a,b)$ and $x_n \to x$, imply $x \in D(t)$. Actually we are using only $t_n \uparrow t$. ■

In view of (1.6) it is easy to check that (A5) is equivalent to

(A5)′ $\liminf_{h \downarrow 0} h^{-1} d\left(S(h)x + \int_t^{t+h} S(t+h-s)F(t,x)ds; D(t+h)\right) = 0$

for all $t \in (a,b)$ and $x \in D(T)$.

We shall prove that (A1)–(A5) guarantee the local existence (and uniqueness) of the solution to (1.5). For the global existence additional hypotheses are needed, namely:

(B1) The mapping $t \to D(t)$ from $(a, b)$ to $2^X$ is closed.

(B2) For each $s \in (a, b)$ and for each connected component $C(s)$ of $D(s)$, there exists a continuous function $w : [s, \tilde{b}] \to X$, such that $w(s) \in C(s)$ and $w(t) \in D(t)$ for all $t \in [s, \tilde{b}]$ (for some $\tilde{b} \in (a, b])$.

REMARK 1.2. Obviously, (B1) implies the latter assertion in (A2). It is well known that (A4) is equivalent to

$$\langle F(t, x_1) - F(t, x_2), x_1 - x_2 \rangle_i \leqq g(t) \| x_1 - x_2 \|^2 \tag{1.7}$$

for all $x_i \in D(t)$, $i = 1, 2$, $t \in (a, b)$ where $\langle y, x \rangle_i = \inf\{x^*(y); x^* \in J(x)\}$ and $J$ is the duality mapping of $X$ (see e.g. [13], [15]).

Clearly, in the case $D(t) = D$, independent of $t$, (A5) becomes

$$\text{(A5)}'' \qquad \liminf_{h \downarrow 0} h^{-1} d(S(h)x + hF(t, x); D) = 0, \qquad \forall t \in (a, b), \quad x \in D.$$

Moreover, if $x \in D \cap D(A)$, then (A5)″ implies

$$\text{(A5)}''' \qquad \liminf_{h \downarrow 0} h^{-1} d(x + h(Ax + F(t, x)); D) = 0.$$

Note that if $x$ is an interior point of $D$ then (A5)″ is satisfied. ■

In the theory of mild solutions to (1.4) the following conditions were used ([8], pp. 350–353):

$$\text{(A6)} \qquad S(t) : D \to D, \qquad \liminf_{h \downarrow 0} h^{-1} d(x + hF(t, x); D) = 0,$$
$$t \in (a, b), \quad x \in D.$$

It is easy to see that (A5)″ is strictly more general than (A6). To prove this fact we first note that (A6) is equivalent to:

(A6)′ For each $t \in (a, b)$ and $x \in D$ there exist $h_n \downarrow 0$ and $x_n \in D$ such that

$$h_n^{-1} \| x + h_n F(t, x) - x_n \| \to 0 \quad \text{as } n \to \infty, \qquad S(t) : D \to D.$$

Set $r_n = (x_n - x - h_n F(t, x))/h_n$. Then $r_n \to 0$ as $n \to \infty$ and $S(h_n)x_n \in D$. We now have

$$h_n^{-1}d(S(h_n)x + h_nF(t,x); D) \leqq h_n^{-1}\|S(h_n)x + h_nF(t,x) - S(h_n)x_n\|$$
$$= \|F(t,x) - S(h_n)F(t,x) - S(h_n)r_n\| \to 0, \qquad \text{as } n \to \infty$$

and therefore (A6) implies (A5)″. The following simple example shows that the converse implication is not true.

Take $X = R$, $D = [1,\infty)$, $y \geqq 1$ and $S(t) = e^{-t}$. Then we have

$$\lim_{h \downarrow 0} h^{-1}d(e^{-h}x + hy; D) = 0 \qquad \text{for all } x \geqq 1.$$

This since $e^{-h} + hy \geqq 1$ for all sufficiently small $h > 0$. However (A6) is not verified, since $e^{-t}$ does not map $[1, +\infty)$ into itself.

We now are able to state our main results:

THEOREM 1.1. *Suppose that* (A1)–(A4) *are fulfilled. Then* (A5) *is a necessary and sufficient condition in order that for each* $t_0 \in (a,b)$ *and* $x \in D(t_0)$ *there exists a local (unique) solution* $u$ *to* (1.5), *on a subinterval* $[t_0, T] \subset [t_0, b)$, *with* $T = T(t_0, x)$. ■

In connection with the extendability of the local solution, the result is given by

THEOREM 1.2. *In addition to* (A1)–(A5) *suppose that* (B1) *and* (B2) *are also fulfilled (for a number* $\tilde{b} \in (a,b]$). *Then* (1.5) *has a unique solution on the entire* $[t_0, \tilde{b})$, *i.e.* $J_u = [t_0, \tilde{b})$. ■

An immediate consequence of Theorem 1.2 is given by

COROLLARY 1.1. *Suppose that* (A1) *holds. Let* $D$ *be a closed subset of* $X$ *and let* $F : (a,b) \times D \to X$ *be a continuous function which satisfies* (A4) *with* $D(t) = D$ *for all* $t \in (a,b)$. *Then for each* $t_0 \in (a,b)$ *and* $x \in D$, *there exists a unique solution* $u : [t_0, b) \to D$ *to* (1.5) *if and only if* (A5)″ *holds.*

Note that the conclusions of the above results were known only under one of the additional conditions below:

(1) $F$ maps bounded subsets into bounded subsets and $S(t): D \to D$ (Martin [8], p. 353).

(2) $S(t): D \to D$ and $F(t,x) = F(x)$ — independent of $t$ ([8], p. 355).

(3) $D = X$ and $F(t,x) = F(x)$ (Webb [19], [11]).

(4) $A = 0$, i.e. $S(t) = I$ (Kenmochi and Takahashi [4], [5], [17]).

(5) $A = 0$ and $D = X$ ([9], [10], [11]).

We have to point out that Pazy [18] assumes the compactness of $S(t)$ for $t > 0$, instead of (A4). His result was extended in many directions (see e.g. [12], [15]).

## 2. Proof of the main results

For the proof of Theorems 1.1 and 1.2 we combine some techniques from [2], [4], [5], [8], [12], and [19]. For proving the uniqueness a new technique is given. The proof of these theorems is quite difficult, so we break it into several parts as follows:

PROPOSITION 2.1. *Suppose that* (A1)–(A3) *and* (A5) *are fulfilled. Let* $t_0 \in (a,b)$, $x \in D(t_0)$, $T \in (t_0, b)$, $r > 0$ *and* $M > 0$ *be such that* $\|F(t,y)\| \leqq M$, $\forall t \in [t_0, T]$, $y \in B(x,r) \cap D(t)$,

$$\sup_{0 \leqq s \leqq T_0} \|S(s)x - x\| + T_0(M+1)N \leqq r, \qquad \text{where } T_0 = T - t_0, \quad N = \exp(\omega T_0).$$

*Then for each positive integer n there exist* $\{t_i^n\}_{i=0}^{\infty}$ *in* $[t_0, T]$ *and a* $n^{-1}$*-approximate solution* $u_n$ *on* $[t_0, T]$ *in the following sense:*

$$t_0^n = t_0, \quad t_i^n < t_{i+1}^n \quad \text{if } t_i^n < T \quad \text{and} \quad t_{i+1}^n = t_i^n \quad \text{if } t_i^n = T,$$

$$d_i^n = t_{i+1}^n - t_i^n \leqq 1/n, \qquad \lim_{i \to \infty} t_i^n = T, \tag{2.1}$$

$$u_n(t_0^n) = x, \qquad x_i^n \equiv u_n(t_i^n) \in D(t_i^n) \cap B(x,r), \tag{2.2}$$

$\|F(t,y) - F(t_i^n, x_i^n)\| \leqq 1/n$, *for all* $t \in [t_i^n, t_{i+1}^n]$, $y \in D(t)$ *with*

$$\|y - x_i^n\| \leqq \sup_{0 \leqq s \leqq d_i^n} \|S(s)x_i^n - x_i^n\| + d_i^n(M+1)N, \tag{2.3}$$

$$u_n(t) = S(t - t_i^n)x_i^n + \int_{t_i^n}^{t} S(t-s)F(t_i^n, x_i^n)ds + (t - t_i^n)p_i^n$$
$$\text{for } t \in [t_i^n, t_{i+1}^n], \quad \text{with } \|p_i^n\| \leqq 1/n. \tag{2.4}$$

PROOF. The construction of $t_i^n$ and $u_n$ is by induction on $i$. Since in this proof there is no danger of confusion we omit $n$ as a superscript for $t_i$, $x_i$ and $d_i$. Set $t_0^n = t_0$ and $x_0^n = x$. Assume that $u_n$ is defined on $[t_0, t_i]$, $t_i < T$ and (2.2)–(2.4) are fulfilled on $[t_0, t_i]$. Let us show the construction of $t_{i+1}$ and $u_n$ on $[t_i, t_{i+1}]$. To this aim define $\delta(x_i)$ as the supremum of all $h$ with the properties:

$$0 < h \leqq 1/n, \qquad t_i + h \leqq T, \tag{2.5}$$

$$\sup_{0 \leqq s \leqq h} \|S(s)z - z\| \leqq 1/n, \qquad \forall z \in \{x_j, F(t_j, x_j), j = 0, 1, \cdots, i\}, \tag{2.6}$$

$$\|F(t,y) - F(t_i, x_i)\| \leqq 1/n, \qquad \forall t \in [t_i, t_i + h], \quad y \in D(t) \tag{2.7}$$
$$\text{with } \|y - x_i\| \leqq \sup_{0 \leqq s \leqq h} \|S(s)x_i - x_i\| + h(M+1)N,$$

$$d\left(S(h)x_i + \int_{t_i}^{t_i+h} S(t_i + h - s)F(t_i, x_i); D(t_i + h)\right) \leqq h/2n. \tag{2.8}$$

Then there exists $h = d_i^n \in (2^{-1}\delta(x_i), \delta(x_i))$ satisfying (2.5)–(2.8). Set $t_{i+1} = t_i + d_i$. For $h = d_i$, (2.8) yields

$$d\left(S(d_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)F(t_i, x_i)ds; D(t_{i+1})\right) \leqq d_i/2n \tag{2.9}$$

which shows that there is $x_{i+1} \in D(t_{i+1})$ of the form

$$x_{i+1} = S(t_{i+1} - t_i)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1} - s)F(t_i, x_i)ds + (t_{i+1} - t_i)p_i \tag{2.10}$$

with $\|p_i\| \leqq 1/n$. Define $u_n$ on $[t_i, t_{i+1}]$ as indicated by (2.4). Introducing the functions $a_n(s) = t_j$ on $[t_j, t_{j+1})$, $j = 0, 1, \cdots, i$, $a_n(T) = T$, one easily checks that $u_n$ can be written in the form

$$u_n(t) = S(t - t_0)x + \int_{t_0}^{t} S(t - s)F(a_n(s), u_n(a_n(s)))ds + g_n(t) \tag{2.11}$$

where

$$g_n(t) = \sum_{j=0}^{i-1} (t_{j+1} - t_j)S(t - t_{j+1})p_j + (t - t_i)p_i, \qquad t \in [t_i, t_{i+1}) \tag{2.11$'$}$$

hence $\|g_n(t)\| \leqq N(t - t_0)/n$ for all $t \in [t_0, t_{i+1}]$.

In view of the induction hypothesis and of the choice of $T$, for $t \in [t_i, t_{i+1}]$ we have

$$\|u_n(t) - x\| \leqq \|S(t - t_0)x - x\| + (t - t_0)MN + N(t - t_0)/n \leqq r.$$

Set $\bar{t} = \lim_{i\to\infty} t_i$. Clearly $\bar{t} \leqq T$. By standard arguments (see [12]) [15], [19] one shows that $\lim_{i\to\infty} x_i = \bar{x}$ exists, so $\bar{x} \in D(\bar{t}) \cap B(x, r)$ (on the basis of (A2)). We have to prove that $\bar{t} = T$. The proof is by contradiction. Hence let $\bar{t} < T$. Choose $c \in (0, 1/n)$ such that $\bar{t} + c < T$ and

$$\|F(\bar{t}, \bar{x}) - F(t, y)\| \leqq 1/4n, \tag{2.12}$$
$$y \in D(t), \quad \|y - \bar{x}\| \leqq 2c + 2c(M+1)N, \quad |t - \bar{t}| \leqq 3c.$$

Let us consider the compact $K$ given by

$$K = \{x_i, F(t_i, x_i), \bar{x}, F(\bar{t}, \bar{x}), i = 1, \cdots\}.$$

There exists $\bar{h} \in (0, c)$ with the properties

$$\sup_{0 \leqq s \leqq 2\bar{h}} \| S(s)z - z \| \leqq c, \qquad \forall z \in K, \tag{2.13}$$

$$d\left(S(\bar{h})\bar{x} + \int_{\bar{t}}^{\bar{t}+\bar{h}} S(\bar{t}+\bar{h}-s)F(\bar{t},\bar{x})ds ; D(\bar{t}+\bar{h})\right) = \bar{h}/4n. \tag{2.14}$$

Set $h_i = \bar{t} + \bar{h} - t_i$. Hence $h_i > \bar{h}$ and $h_i \downarrow \bar{h}$ as $i \to \infty$. Let $i_0$ be a positive integer with the property that

$$h_i < 2\bar{h}, \quad \bar{t} - t_i < \bar{h}, \quad \| x_i - \bar{x} \| \leqq \bar{h}, \quad \delta(x_i) < \bar{h} < h_i, \qquad \forall i \geqq i_0.$$

Note that $\delta(x_i) < 2d_i = 2(t_{i+1} - t_i) \to 0$ as $i \to \infty$. In view of (2.12) and (2.13) it is easy to verify that (2.6) and (2.7) hold with $h_i$ in place of $h$. Since $h_i > \delta(x_i)$ for $i \geqq i_0$, it follows that for $h = h_i$, (2.8) is not true, i.e.

$$d\left(S(h_i)x_i + \int_{t_i}^{t_i+h_i} S(t_i + h_i - s)F(t_i, x_i)ds ; D(t_i + h_i)\right) > h_i/2n \quad \text{for all } i \geqq i_0. \tag{2.15}$$

Letting $i \to \infty$ in (2.15) one obtains an inequality which contradicts (2.14) (one observes also that $t_i + h_i = \bar{t} + \bar{h}$). On the basis of (2.4), $\lim_{t \uparrow T} u_n(t) = \lim_{i \to \infty} x_i = \bar{x}$, so defining $u_n(T) = \bar{x}$, completes the proof.

REMARK 2.1. For $y = x_{i+1}$, (2.3) yields

$$\| F(t_{i+1}^n, x_{i+1}^n) - F(t_i^n, x_i^n) \| \leqq 1/n, \qquad i = 0, 1, \cdots. \tag{2.16}$$

PROPOSITION 2.2. *In addition to the hypotheses of Proposition* 2.1 *suppose that* $\lim_{n \to \infty} u_n(t) = u(t)$ *exists uniformly on* $[t_0, T]$. *Then* $u$ *is a solution to* (1.5) *on* $[t_0, T]$.

PROOF. Let $t \in [t_0, T)$. Then for each $n$ there exists $i = i_n(t) = i_n$ such that $t \in [t_i, t_{i+1})$, hence $|a_n(t) - t| \leqq |t - t_i| \leqq 1/n$. It follows that $u_n(a_n(t)) \to u(t)$ as $n \to \infty$ uniformly on $[t_0, T]$. On the other hand, $u_n(a_n(t)) = x_{i_n}^n \in D(t_{i_n}^n) \cap B(x, r)$ and $t_{i_n}^n \to t$, $x_{i_n}^n \to u(t)$ as $n \to \infty$. On the basis of (A2) we have $u(t) \in D(t) \cap B(x, r)$. Finally, letting $n \to \infty$ in (2.11), the result follows. ■

For the proof of Theorem 2.1 some other results are needed. Let $m$ and $n$ be positive integers and let $U = \{t_i^n, t_j^m; i, j = 0, 1, \cdots\}$. Denote by $\{r_e\}_{e=0}^{\infty}$ the minimal refinement of the partitions $\{t_i^n\}$ and $\{t_j^m\}$ of $[t_0, T]$, i.e.

$$r_0 = t_0, \quad r_{e+1} = \min\{t \in U; t > r_e\}, \qquad e = 0, 1, \cdots.$$

PROPOSITION 2.3. *Assume that the hypotheses of Proposition* 2.1 *hold. Then there exist the functions* $v_n$ *and* $v_m$ *from* $[t_0, T]$ *into* $X$ *with the following properties*:

*For each positive integer e, there exists a partition $\{s^e_k\}_{k=0}^{\infty}$ of $[r_e, r_{e+1}]$ such that*

$$s_0 = r_e, \quad s_{k+1} = s_k \quad \textit{if } s_k = r_{e+1}, \quad 0 < s_{k+1} - s_k \equiv h_k \leqq r_{e+1} - r_e \leqq \min\{1/n, 1/m\},$$

(2.17)

$$\lim_{k\to\infty} s_k = r_{e+1}$$

*where e as a superscript for $s_k$ is omitted. Furthermore*

$$v_p(t_0) = x, \quad v^p_k = v^p_{k,e} = v_p(s_k) \in D(s_k) \cap B(x, r),$$

(2.18)

$$\lim_{s \uparrow r_{e+1}} v_p(s) = v_p(r_{e+1} -) \in D(r_{e+1}) \cap B(x, r), \quad p = m, n; \quad e = 0, 1, \cdots.$$

*If for the integers i and j we have the situation*

$$t^n_i, t^m_j \leqq r_e < r_{e+1} \leqq t^n_{i+1}, t^m_{j+1},$$

*then*:

$$\text{(2.19)} \quad \begin{cases} (1) \ v_n(r_e) = u_n(r_e) \ \textit{if } r_e = t^n_i, \textit{ and } v_n(r_e) = v_n(r_e -) \ \textit{if } r_e = t^m_j, \\ (2) \ v_m(r_e) = u_m(r_e) \ \textit{if } r_e = t^m_j, \textit{ and } v_m(r_e) = v_m(r_e -) \ \textit{if } r_e = t^n_i. \end{cases}$$

*Moreover, for all $t \in [r_e, r_{e+1}]$ we have*

$$\text{(2.20)} \quad \|v_n(t) - u_n(t)\| \leqq 3(t - t^n_i)N/n, \quad \|v_m(t) - u_m(t)\| \leqq 3(t - t^m_j)N/m.$$

*Furthermore*

$$\text{(2.21)} \quad \|v_p(t^p_{i+1}) - v_p(t^p_{i+1} -)\| \leqq 3(t^p_{i+1} - t^p_i)N/p, \qquad p = n, m; \quad i = 0, 1, \cdots,$$

$$\|F(t, y) - F(s_k, v^p_k)\| \leqq 1/p, \quad t \in [s_k, s_{k+1}], \qquad \forall y \in D(t)$$

(2.22)

$$\textit{with} \quad \|y - v^p_k\| = \sup_{0 \leqq s \leqq h_k} \|S(s)v^p_k - v^p_k\| + h_k(M+1)N, \qquad p = m, n,$$

$$\sup_{0 \leqq s \leqq h_k} \|S(s)z - Z\| \leqq \min\{1/n, 1/m\},$$

(2.23)

$$\forall z \in \{v^p_q, F(s_q, v^p_q), \ q = 0, 1, \cdots; \ p = m, n\},$$

$$v^p(t) = S(t - s_k)v^p_k + \int_{s_k}^t S(t-s)F(s_k, v^p_k)ds + (t - s_k)q^{p,e}_k$$

(2.24)

$$\textit{with} \quad \|q^{p,e}_k\| \leqq 1/p, \quad r_e \leqq s_k \leqq t < s_{k+1} \leqq r_{e+1}, \quad p = m, n$$

*(so $v_p$ is continuous on $[r_e, r_{e+1})$). If in addition to the above hypotheses we suppose that* (A4) *is also fulfilled, then the following estimate holds*:

$$\begin{aligned}&\| v_n(r_e-)-v_m(r_e-)\| \\ &\leqq 6NT_0(m^{-1}+n^{-1})\exp(2g(T)T_0+\omega T_0), \quad e=0,1,\cdots.\end{aligned} \tag{2.25}$$

PROOF. Let $e$ be a positive integer. Assume that $v_m$ and $v_n$ are constructed on $[t_0, r_e]$ with the above properties on $[t_0, r_e]$. We now show the construction of $v_m$ and $v_n$ on $[r_e, r_{e+1}]$.

Set $s_0 = r_e$ and define $v_0^n = v_n(s_0)$ and $v_0^m = v_m(s_0)$ as indicated by (2.19). Suppose that $v_n$ and $v_m$ are defined on $[s_0, s_k]$ by (2.24) with $k-1$ in place of $k$. Define $v_n$ and $v_m$ on $[s_k, s_{k+1}]$ as follows: if $s_k = r_{e+1}$ set $s_{k+1} = r_{e+1}$ and if $s_k < r_{e+1}$ set $s_{k+1} = s_k + h_k$, where $h_k$ is defined below:

Let $\delta_k$ be the supremum of all $h$ with the properties

$$\begin{aligned}&0<h\leqq \min\{m^{-1},n^{-1}\}, \quad s_k+h\leqq r_{e+1}, \quad \sup_{0\leqq s\leqq h}\|S(s)z-z\|\leqq \min\{m^{-1},n^{-1}\} \\ &\qquad \forall z\in\{v_j^p, F(s_j,v_j^p), j=0,1,\cdots,k;\ p=m,n\},\end{aligned} \tag{2.26}$$

$$\|F(t,y)-F(s_k,v_k^p)\|\leqq 1/p, \quad \forall t\in[s_k,s_k+h], \quad \forall y\in D(t) \tag{2.27}$$

with

$$\|y-v_k^p\|\leqq \sup_{0\leqq s\leqq h}\|S(s)v_k^p-v_k^p\|+h(M+1)N, \qquad p=m,n$$

$$d\left(S(h)v_k^p+\int_{s_k}^{s_k+h}S(s_k+h-s)F(s_k,v_k^p)ds; D(s_k+h)\right)\leqq h/2p, \ p=m,n. \tag{2.28}$$

We now choose $h_k\in(2^{-1}\delta_k,\delta_k]$ satisfying (2.26)–(2.28) with $h_k$ in place of $h$. Substituting $h=h_k$ into (2.28) we conclude that there exists $v_{k+1}^p\in D(s_k+h_k)\equiv D(s_{k+1})$ such that

$$v_{k+1}^p = S(s_{k+1}-s_k)v_k^p+\int_{s_k}^{s_{k+1}}S(s_{k+1}-s)F(s_k,v_k^p)ds+(s_{k+1}-s_k)q_k^{p,e} \tag{2.29}$$

with $\|q_k^{p,e}\|\leqq 1/p$. For simplicity set $q_k^{p,e}=q_k^p$, $p=m,n$.

We now define $v_p$ on $[s_k, s_{k+1})$ as indicated by (2.24). Similarly to $u_n$ given by (2.11), one checks that $v_p$ can be written under the form

$$v_p(t)=S(t-s_0)v_0^p+\int_{s_0}^{t}S(t-s)F(b(s),v_p(b(s)))ds+q_e^p(t) \tag{2.30}$$

for $s_0\leqq t<s_{k+1}$, where $b(s)=s_j$ on $[s_j,s_{j+1})$, $j=0,1,\cdots,k$ and

$$q_e^p(t)=\sum_{j=1}^{k-1}(s_{j+1}-s_j)S(t-s_{j+1})q_j^p+(t-s_k)q_k^p, \qquad t\in[s_k,s_{k+1}], \tag{2.31}$$

hence $\|q_e^p(t)\| \leqq N(t-s_0)/p$, $p=m,n$.

Suppose that the situation in (2.19) is the following one:

$$t_i^n < t_j^m = r_e < r_{e+1} = t_{i+1}^n < t_{j+1}^m. \tag{2.32}$$

Then by the convention (2.19)

$$s_0^e \equiv s_0 = r_e = t_j^m, \quad v_0^n \equiv v_n(t_j^m) = v_n(t_j^m -), \quad v_0^m \equiv v_m(t_j^m) \equiv x_j^m. \tag{2.33}$$

Accordingly, for $p=m,n$, (2.30) becomes respectively

$$v_m(t) = S(t-t_j^m)x_j^m + \int_{t_j^m}^t S(t-s)F(b(s), v_m(b(s)))ds + q_e^m(t) \tag{2.34}$$

with

$$\|q_e^m(t)\| \leqq N(t-t_j^m)/m, \qquad t \in [t_j^m, s_{k+1}),$$

$$v_n(t) = S(t-t_j^m)v_n(t_j^m -) + \int_{t_j^m}^t S(t-s)F(b(s), v_n(b(s)))ds + q_e^n(t) \tag{2.35}$$

with

$$\|q_e^n(t)\| \leqq N(t-t_j^m)/n, \qquad t \in [t_j^m, s_{k+1}).$$

On the other hand, by (2.11) (with $m$ in place of $n$) we have

$$u_m(t_j^m) = x_j^m = S(t_j^m - t_0)x + \int_{t_0}^{t_j^m} S(t_j^m - s)F(a_m(s), u_m(a_m(s)))ds + g_m(t_j^m) \tag{2.36}$$

with

$$\|g_m(t_j^m)\| \leqq N(t_j^m - t_0)/m.$$

Substituting $x_j^m$ into (2.34) one obtains

$$\begin{aligned} v_m(t) = {} & S(t-t_0)x + \int_{t_0}^{t_j^m} S(t-s)F(a_m(s), u_m(a_m(s)))ds \\ & + \int_{t_j^m}^t S(t-s)F(b(s), v_m(b(s)))ds + S(t-t_j^m)g_m(t_j^m) + q_e^m(t) \end{aligned} \tag{2.37}$$

with

$$\|S(t-t_j^m)g_m(t_j^m) + q_e^m(t)\| \leqq N(t-t_0)/m, \quad t \in [t_j^m, s_{k+1}) \text{ (see (2.11)')}.$$

By the induction hypothesis, $v_m(b(s)) \in D(b(s)) \cap B(x,r)$ for $s \in [s_0, s_{k+1})$, hence $\|F(b(s), v_m(b(s)))\| \leqq M$ on $[s_0, s_{k+1})$. Thus (2.37) yields

$$\|v_m(t) - x\| \leqq \|S(t-t_0)x - x\| + (t-t_0)(M+1)N = r, \qquad t \in [t_0, s_{k+1}).$$

Similarly one shows that $\|v_n(t)-x\| \leqq r$ on $[t_0, s_{k+1})$. Arguing as for the proof of: $\lim_{i\to\infty} t_i^n = T$ (see (2.12)–(2.15)) one proves that $s_k \uparrow r_{e+1}$ as $k \to \infty$. The details are left to the reader (it can be found in the book [15], ch. 5). Moreover one shows that $\lim_{k\to\infty} v_p(s_k) \equiv x_p$ exists, and in view of $v_p(s_k) \in D(s_k) \cap B(x,r)$ it follows that $x_p \in D(r_{e+1}) \cap B(x,r)$, $p = m, n$.

Finally, on the basis of (2.24) (where $v_k^p = v_p(s_k)$) we have

$$\|v_p(t) - v_k^p\| \leqq \|(S(t-s_k)-I)v_k^p\| + (t-s_k)(M+1)N, \qquad t \in [s_k, s_{k+1}),$$

which implies

$$v_p(r_{e+1}-) = \lim_{t \uparrow r_{e+1}} v_p(t) = \lim_{k\to\infty} v_k^p \equiv x_p \in D(r_{e+1}) \cap B(x,r).$$

Consequently, it remains to prove (2.20), (2.21) and (2.25).

In the situation (2.32), $v_m$ is given by (2.34) on $[r_e, r_{e+1})$, while $u_m$ has the form (2.24) with $m$ in place of $n$, i.e.

$$u_m(t) = S(t-t_j^m)x_j^m + \int_{t_j^m}^t S(t-s)F(t_j^m, x_j^m)ds + (t-t_j^m)p_j^m,$$

$$\text{(2.38)} \qquad t \in [t_j^m, t_{j+1}^m], \quad \|p_j^m\| \leqq 1/m.$$

Therefore we have

$$\|v_m(t)-u_m(t)\| \leqq \int_{t_j^m}^t N\|F(b(s), v_m(b(s))) - F(t_j^m, x_j^m)\|ds + 2(t-t_j^m)N/n,$$

$$\text{(2.39)} \qquad t \in [t_j^m, t_{i+1}^n) \quad \text{(see (2.32))}.$$

In view of (2.34),

$$\|v_m(b(s)) - x_j^m\| \leqq \sup_{0\leqq s \leqq d_j} \|S(s)x_j^m - x_j^m\| + d_j^m(M+1)N, \qquad s \in [t_j^m, t_{i+1}^n),$$

where $d_j^m = t_{j+1}^m - t_j^m$. By the definition of $d_j = d_j^m$, (2.7) yields

$$\text{(2.40)} \qquad \|F(b(s), v_m(b(s))) - F(t_j^m, x_j^m)\| \leqq 1/m, \qquad s \in [t_j^m, t_{i+1}^n)$$

and thus $\|v_m(t)-u_m(t)\| \leqq 3(t-t_j^m)N/n$, on $[t_j^m, t_{i+1}^n]$.

In the situation (2.32) the following two possibilities may occur:

(1°) either $t_i^n < t_{j-1}^m$ (so $r_{e-1} = t_{j-1}^m$), or

(2°) $t_{j-1}^m < t_i^n$.

In the case (1°), (2.30) gives

$$\text{(2.41)} \quad v_m(t) = S(t-t_{j-1}^m)x_{j-1}^m + \int_{t_{j-1}^m}^t S(t-s)F(b(s), v_m(b(s)))ds + q_{e-1}^m(t)$$

for $t \in [t^m_{j-1}, t^m_j)$, where $\|q^m_{e-1}(t)\| \leqq (t - t^m_{j-1})N/m$, hence

$$v_m(t^m_j -) = S(t^m_j - t^m_{j-1})x^m_{j-1} + \int_{t^m_{j-1}}^{t^m_j} S(t^m_j - s)F(b(s), v_m(b(s)))ds + q^m_{e-1}(t^m_j) \tag{2.42}$$

with $\|q^m_{e-1}(t^m_j)\| \leqq (t^m_j - t^m_{j-1})N/m$. Inasmuch as

$$\begin{aligned} v_m(t^m_j) &= u_m(t^m_j) \\ &= S(t^m_j - t^m_{j-1})x^m_{j-1} + \int_{t_{j-1}}^{t^m} S(t^m_j - s)F(t^m_{j-1}, x^m_{j-1})ds + (t^m_j - t^m_{j-1})p^m_{j-1}, \\ &\qquad \|p^m_{j-1}\| \leqq 1/m \end{aligned} \tag{2.43}$$

and arguing as for (2.40) (with $j-1$ and $e$ in place of $j$ and $e+1$, respectively), (2.42) and (2.43) yield

$$\|v_m(t^m_j -) - v_m(t^m_j)\| \leqq 3(t^m_j - t^m_{j-1})N/m. \tag{2.44}$$

We now prove that (2.44) holds in the case (2°) too. Let us observe that in this case (i.e. $t^m_{j-1} < t^n_i$ and (2.32)), $r_{e-1} = t^n_i$. However, we do not necessarily have $r_{e-2} = t^m_{j-1}$, since the following situation may occur:

(3°) $t^m_{j-1} < t^n_{i-k} \leqq t^n_i$, $k = 0, 1, \cdots, q$.

The calculus below allows us to see that we may consider only $q = 0$, i.e.

$$t^n_{i-1} \leqq t^m_{j-1} < t^n_i \quad (\text{so } r_{e-2} = t^m_{j-1},\ r_{e-1} = t^n_i,\ r_e = t^m_j,\ r_{e+1} = t^n_{i+1} < t^m_{j+1}). \tag{2.45}$$

Clearly, (2.30) holds for all $t \in [r_e, r_{e+1})$. In the case of the intervals $[r_{e-1}, r_e)$ and $[r_{e-2}, r_{e-1})$, (2.30) yields respectively

$$v_m(t^m_j -) = S(t^m_j - t^n_i)v_m(t^n_i) + \int_{t^n_i}^{t^m_j} S(t^m_j - s)F(b(s), v_m(b(s)))ds + q^m_{e-1}(t^m_j) \tag{2.46}$$

with $\|q^m_{e-1}(t^m_j)\| \leqq (t^m_j - t^n_i)/m$,

$$\begin{aligned} v_m(t^n_i -) &= S(t^n_i - t^m_{j-1})v_m(t^m_{j-1}) + \int_{t^m_{j-1}}^{t^n_i} S(t^n_i - s)F(b(s), v_m(b(s)))ds + q^m_{e-2}(t^n_i), \\ &\qquad \|q^m_{e-2}(t^n_i)\| \leqq (t^n_i - t^m_{j-1})N/m. \end{aligned} \tag{2.47}$$

On the other hand, $v_m(t^n_i) = v_m(t^n_i -)$ and $v_m(t^m_{j-1}) = x^m_{j-1}$, so substituting $v_m(t^n_i -)$ into (2.46) one obtains

$$
\begin{aligned}
v_m(t_j^m-) = S(t_j^m - t_{j-1}^m)x_{j-1}^m &+ \int_{t_{j-1}^m}^{t_i^n} S(t_j^m - s)F(b(s), v_m(b(s)))ds \\
&+ S(t_j^m - t_i^n)q_{e-2}^m(t_i^n) \\
&+ \int_{t_{j-1}^n}^{t_j^m} S(t_j^m - s)F(b(s), v_m(b(s)))ds + q_{e-1}^m(t_j^m).
\end{aligned} \tag{2.48}
$$

Let us write the formula (2.43) under the form

$$
\begin{aligned}
v_m(t_j^m) = S(t_j^m - t_{j-1}^m)x_{j-1}^m &+ \int_{t_{j-1}^m}^{t_i^n} S(t_j^m - s)F(t_{j-1}^m, x_{j-1}^m)ds \\
&+ \int_{t_i^n}^{t_j^m} S(t_j^m - s)F(t_{j-1}^m, x_{j-1}^m)ds + (t_j^m - t_{j-1}^m)p_{j-1}^m.
\end{aligned} \tag{2.49}
$$

Finally, (2.40), (2.48) and (2.49) show that (2.44) holds and in the case (2°) above a quick check of the proof shows that the general case (3°) can be treated in the same manner. We now suppose that (A4) holds and prove the crucial estimate (2.55). Since $v_{k+1}^p = v_p(s_{k+1}) \in D(s_{k+1})$, $p = m, n$ (by (2.18)), then (A4) implies (for $\lambda = h_k = s_{k+1} - s_k$ and $t = s_{k+1}$)

$$
\begin{aligned}
&(1 - g(s_{k+1})h_k)\| v_m(s_{k+1}) - v_n(s_{k+1})\| \\
&\leqq \| v_m(s_{k+1}) - v_n(s_{k+1}) - h_k(F(s_{k+1}, v_m(s_{k+1})) - F(s_{k+1}, v_n(s_{k+1})))\|.
\end{aligned} \tag{2.50}
$$

Combining (1.3), (2.29), (2.50) and the elementary inequality

$$(1-t)^{-1} \leqq \exp 2t, \qquad t \in [0, \tfrac{1}{2}] \tag{2.50$'$}$$

we get

$$
\begin{aligned}
\| v_{k+1}^m - v_{k+1}^n \| \leqq \Big[ \| v_k^m - v_k^n \| &+ \int_{s_k}^{s_{k+1}} \| S(s_{k+1} - s)F_k^m - F_{k+1}^m \| ds + \frac{h_k}{n} + \frac{h_k}{m} \\
&+ \int_{s_k}^{s_{k+1}} \| S(s_{k+1} - s)F_k^n - F_{k+1}^n \| ds \Big] \exp(2g(T) + \omega)h_k
\end{aligned} \tag{2.51}
$$

where $F_k^p = F(s_k, v_k^p)$, $p = m, n$, and $g(T)h_k \leqq \frac{1}{2}$.

On the basis of (2.23), (2.26), (2.27) and (2.29) it follows that

$$\| F_{k+1}^p - F_k^p \| \leqq 1/p, \quad \| S(s_{k+1} - s)F_k^p - F_k^p \| \leqq 1/p, \quad s \in [s_k, s_{k+1}] \tag{2.52}$$

and therefore (2.51) implies

$$\| v_{k+1}^m - v_{k+1}^n \| \leqq [\| v_k^m - v_k^n \| + 3(m^{-1} + n^{-1})h_k]\exp(2g(T) + \omega)h_k. \tag{2.53}$$

Substituting $h_k = s_{k+1} - s_k$ into (2.53) and iterating, we get

$$\begin{aligned}(2.54)\qquad \|v_m(s_{k+1})-v_n(s_{k+1})\| &\leqq [\|v_m(s_0)-v_n(s_0)\|+3(m^{-1}+n^{-1})(s_{k+1}-s_0)] \\ &\quad \times \exp(2g(T)+\omega)(s_{k+1}-s_0).\end{aligned}$$

Letting $k\to\infty$ (i.e. $s_{k+1}\uparrow r_{e+1}$), (2.54) yields ($e=0,1,\cdots$)

$$\begin{aligned}(2.55)\qquad \|v_m(r_{e+1}-)-v_n(r_{e+1}-)\| &\leqq [\|v_m(r_e)-v_n(r_e)\|+3(m^{-1}+n^{-1})(r_{e+1}-r_e)] \\ &\quad \times \exp(2g(T)+\omega)(r_{e+1}-r_e).\end{aligned}$$

Let us observe that

$$\begin{aligned}(2.56)\qquad \|v_m(r_e)-v_n(r_e)\| &\leqq \|v_m(r_e-)-v_n(r_e-)\|+\|v_m(r_e-)-v_m(r_e)\| \\ &\quad +\|v_n(r_e-)-v_m(r_e)\|.\end{aligned}$$

Taking into account (2.21)

$$(2.57)\qquad \sum_{k=0}^{e}\|v_p(r_k)-v_p(r_k-)\|\leqq 3NT_0/p, \qquad p=m,n.$$

Substituting (2.56) and (2.57) into (2.55) and then iterating (2.55) for $e=0,1,\cdots,e-1$, with $v_m(r_0-)=v_n(r_0-)=x$ (since $t_0=r_0$) one obtains (2.25).

Proof of Theorem 1.1. *Necessity*. Let $t_0\in(a,b)$ and $x\in D(t_0)$. If (1.5) has a solution $u=u(t;t_0,x)$, then

$$u(t_0+h)=S(h)x+\int_{t_0}^{t_0+h}S(t_0+h-s)F(s,u(s))ds\in D(t_0+h)$$

for all sufficiently small $h>0$. Consequently,

$$\begin{aligned}(2.58)\qquad &\lim_{h\downarrow 0} h^{-1}d(S(h)x+hF(t_0,x);D(t_0+h)) \\ &\leqq \lim_{h\downarrow 0} h^{-1}\|S(h)x+hF(t_0,x)-u(t_0+h)\| \\ &=\lim_{h\downarrow 0}\left\|F(t_0,x)-\frac{1}{h}\int_{t_0}^{t_0+h}S(t_0+h-s)F(s,u(s))ds\right\| \\ &=0\end{aligned}$$

so (A5) follows even with "lim" in place of "lim inf"

*Sufficiency*. The conditions (A1)–(A3) and (A5) imply the existence of $u_n$, $v_n$ satisfying (2.1)–(2.4), (2.17)–(2.24). We now prove that the dissipativity condition (A4) (which yields (2.25)) ensures both the existence (i.e. the convergence of $u_n$)

and the uniqueness of the solution to (1.5). For proving the convergence of $u_n$ let $t$ be arbitrarily chosen in $[t_0, T]$. Then for each pair $(m, n)$ of positive integers, there are $i = i(t, n)$ and $j = j(t, m)$ such that

$$t \in [t_i^n, t_{i+1}^n] \cap [t_j^m, t_{j+1}^m] \equiv [r_e, r_{e+1}], \qquad \text{with } e = e(t, m, n).$$

Let $k = k(t, m, n)$ be such that $s_k \leqq t < s_{k+1}$. Then by (2.20),

$$\|u_n(t) - v_n(t)\| \leqq 3(t - t_i^n)N/n \leqq 3N/n^2, \quad \|u_m(t) - v_m(t)\| \leqq 3N/m^2. \tag{2.59}$$

On the other hand (2.24) gives

$$\begin{aligned} \|v_p(t) - v_p(s_{k+1})\| &\leqq N\|S(s_{k+1} - t)v_k^p - v_k^p\| \\ &\quad + N\int_{s_k}^{t} \|S(s_{k+1} + t)F(s_k, v_k^p) - F(s_k, v_k^p)\|\, ds \\ &\quad + \left\|\int_t^{s_{k+1}} S(s_{k+1} - s)F(s_k, v_k^p)ds\right\| + \|(s_{k+1} - t)q_k^{p,e}\| \\ &\leqq N/p + N(t - s_k)/p + N(s_{k+1} - t)/p + NM(s_{k+1} - t) \\ &\leqq 2MN(1/p + 1/p^2), \qquad p = m, n \end{aligned} \tag{2.60}$$

where (2.52) was also used. Substituting (2.25) and (2.57) into (2.56) we get

$$\|v_m(r_e) - v_n(r_e)\| \leqq 9CT_0(m^{-1} + n^{-1})\exp(2g(T) + \omega)T_0 \quad \text{with } C = 2MN. \tag{2.61}$$

Going back to (2.54) (in which $s_0 = r_e$) we conclude that

$$\|v_m(s_{k+1}) - v_n(s_{k+1})\| \leqq 12CT_0(m^{-1} + n^{-1})\exp(4g(T) + 2\omega)T_0. \tag{2.62}$$

Finally, a simple combination of (2.60) and (2.62) yields

$$\|v_m(t) - v_n(t)\| \leqq C(m^{-1} + n^{-1})(2 + 12T_0)\exp(4g(T) + 2\omega)T_0$$

which shows that $v_n(t)$ is uniformly Cauchy on $[t_0, T]$. By (2.59) it follows that $u_n(t)$ is also uniformly Cauchy on $[t_0, T]$. Set $u(t) = \lim_{n\to\infty} u_n(t)$. According to Proposition 2.2 $u$ is a solution to (1.5).

*The uniqueness.* In general a mild solution may not be differentiable (see e.g. [8], p. 345). Consequently, for the proof of the uniqueness we cannot use the standard method involving a lemma of Kato ([4], [5], [8, p. 232], [13, pp. 148, 150]). In what follows we give a simple technique, which is independent of both differential inequalities and of integral solutions.

Let $u_1$ and $u_2$ be solutions to (1.5) on $[t_0, T]$, with $u_1(t_0) = x$, and $u_2(t_0) = y$. Then by (A4)

$$(2.63)\quad \begin{aligned}&(1-g(t+h)h)\|u_1(t+h)-u_2(t+h)\|\\ &\leqq \|u_1(t+h)-u_2(t+h)-hF(t+h,u_1(t+h))+hF(t+h,u_2(t+h))\|\end{aligned}$$

for all $t\in[t_0,T)$ with $T=\min\{T_1(t_0,x),T_2(t_0,y)\}$ and for all $h>0$ with $t+h<T$. Clearly, $u_i$ can be written under the form

$$(2.64)\qquad u_i(t+h)=S(h)u_i(t)+\int_t^{t+h}S(t+h-s)F(s,u_i(s))ds,\qquad i=1,2.$$

Combining (1.3), (2.63), (2.64) and the inequality

$$(2.64)'\qquad (1-t)^{-1}\leqq \exp(1+\varepsilon)t,\quad t\in\left[0,\frac{\varepsilon}{1+\varepsilon}\right],\qquad \varepsilon>0$$

we get

$$(2.65)\quad \|u_1(t+h)-u_2(t+h)\|\leqq(\|u_1(t)-u_2(t)\|+\|I_0\|)\exp[(1+\varepsilon)g(t+h)+\omega]h$$

where $hg(T)(1+\varepsilon)\leqq\varepsilon$, and

$$(2.66)\quad \begin{aligned}I_0=\int_t^{t+h}&S(t+h-s)(F(s,u_1(s))-F(s,u_2(s)))\\ &-(F(t+h,u_1(t+h))-F(t+h,u_2(t+h)))ds.\end{aligned}$$

Set $f(t)=F(t,u_1(t))-F(t,u_2(t))$. Let $K$ be a compact of $X$ such that $f(t)\in K$ for all $t\in[t_0,T]$. There is $d=d(\varepsilon)>0$ such that

$$(2.67)\qquad \|S(s)z-w\|\leqq\varepsilon,\qquad \forall s\in(0,d),\quad \|z-w\|\leqq d,\quad z,w\in K.$$

Choose $r=r(\varepsilon)$ with the property

$$\|f(t)-f(s)\|\leqq d,\qquad \forall t,s\in[t_0,T],\quad |t-s|\leqq r.$$

Note that $s\in[t,t+h]$ implies $0\leqq t+h-s\leqq h$, hence

$$(2.68)\qquad \|f(s)-f(t+h)\|\leqq d,\qquad \forall s\in[t,t+h],\quad 0<h<r$$

and therefore by (2.66) and (2.67) we have

$$\|I_0\|\leqq h\varepsilon,\qquad \forall h<d_0(\varepsilon)=\min\{d,r\},\quad \forall t\in[t_0,T),\quad t+h<T.$$

Accordingly, (2.65) yields

$$(2.69)\ \|u_1(t+h)-u_2(t+h)\|\leqq(\|u_1(t)-u_2(t)\|+h\varepsilon)\exp[(1+\varepsilon)g(t+h)+\omega]h$$

$$\text{for all } t\in[t_0,T) \text{ and } h\leqq h_0=\min\{d_0(\varepsilon),\varepsilon/(1+\varepsilon)g(T)\}.$$

Fix $t \in [t_0, T]$ and choose a partition $t_0 < t_1 < \cdots < t_n = t$ of $[t_0, t]$ with $t_k - t_{k-1} = h_k \leqq h_0$. Then (2.69) gives

$$\|u_1(t_k) - u_2(t_k)\| \leqq (\|u_1(t_{k-1}) - u_2(t_{k-1})\| + \varepsilon h_k)\exp[(1+\varepsilon)g(t_k) + \omega]h_k. \tag{2.70}$$

Iterating (2.70) for $k = 1, 2, \cdots, n$, one obtains

$$\|u_1(t) - u_2(t)\| \leqq (\|x - y\| + (t - t_0)\varepsilon)\exp[(1+\varepsilon)g(t) + \omega](t - t_0). \tag{2.70$'$}$$

Since $\varepsilon > 0$ was arbitrarily fixed we conclude that

$$\|u_1(t) - u_2(t)\| \leqq \|x - y\|\exp(g(t) + \omega)(t - t_0), \qquad t \in [t_0, T] \tag{2.71}$$

with $u_1(t_0) = x$, $u_2(t_0) = y$, $x, y \in D(t_0)$. In particular (2.71) implies the uniqueness of the solution to (1.5), which completes the proof.

REMARK 2.2. We now sketch another method (indicated by the referee) for proving the inequality (2.71).

With $t$ in place of $t + h$, (2.64) becomes

$$\begin{aligned} u_i(t) &= S(h)u_i(t-h) + \int_{t-h}^{t} S(t-s)F(s, u_i(s))ds \\ &= S(h)u_i(t-h) + hF(t, u_i(t)) + hr_i(h), \qquad i = 1, 2, \end{aligned} \tag{2.72}$$

where $r_i(h) \to 0$ as $h \downarrow 0$ (by the Lebesgue theorem) and $t \in (t_0, T]$. Combining (A4) (with $\lambda = h$) and (2.72) we get

$$\begin{aligned} (1 - hg(t))\|u_1(t) - u_2(t)\| &\leqq \|u_1(t) - u_2(t) - h(F(t, u_1(t)) - F(t, u_2(t)))\| \\ &\leqq \|u_1(t-h) - u_2(t-h)\|\exp(\omega h) + h(\|r_1(h)\| + \|r_2(h)\|) \end{aligned} \tag{2.73}$$

Obviously, (2.73) can be written in the form

$$\begin{aligned} &-h^{-1}(\|u_1(t-h) - u_2(t-h)\| - \|u_1(t) - u_2(t)\|) \\ &\leqq (g(t) + (\exp(\omega h) - 1)/h)\|u_1(t) - u_2(t)\|. \end{aligned}$$

Letting $h \downarrow 0$ one obtains

$$D_-\|u_1(t) - u_2(t)\| \leqq (g(t) + \omega)\|u_1(t) - u_2(t)\| \tag{2.74}$$

where

$$D_-f(t) = \limsup_{h \downarrow 0} ((f(t-h) - f(t))/(-h)).$$

Solving the differential inequality (2.74) we get (2.71).

Note that the usage of (2.64)′ with $0<\varepsilon<1$ instead of $\varepsilon=1$, may improve some estimates in several papers on difference approximation of Cauchy problems (see e.g. [6], [7], [16]).

REMARK 2.3. The proof of Theorem 1.2 follows a technique of Kenmochi and Takahashi [5], and it is given in [15].

It is well known that for an arbitrary semigroup $S$ of class $C_0$ we have $\|S(t)\|_{L(X)} \leqq M\exp(\omega t)$, $t \geqq 0$. A simple check of the proof shows that for $M>1$ our estimates will not be useful, since the powers of $M$ will appear in the right hand side (referee's remark). However, the conclusion of Theorems 1.1 and 1.2 remains valid for $S$, too. In this case the proof is more complicated and follows some techniques from [17]. Actually, for most applications to PDE, it suffices to restrict ourselves to semigroups of type $\omega$ (see (1.3)). Recall that the semigroups of type 0 are called the contraction semigroups.

COROLLARY 2.1. *Let $D$ be a closed subset of $X$ and let $S$ be a semigroup of type $\omega$. Suppose that $F:(a,b)\times D\to X$ is a continuous function and for each $t\in(a,b)$, the operator $x\to F(t,x)$ is $g(t)$ dissipative (i.e.* (A4) *holds with $D(t)=D$). Then for every $t_0\in(a,b)$ and $x\in D$, there is a unique solution $u:[t_0,b)\to D$ if and only if* (A5)″ *holds.*

PROOF. We observe that for $x\in D$, (B2) holds with $w(t)=x$, $\forall t\in(t_0,\tilde{b})$ and $\tilde{b}=b$. Accordingly, on the basis of Theorem 1.2 the result follows.

## 3. Applications to some partial differential equations

Let $\Omega$ be a bounded domain in $R^m$ with smooth boundary $\Gamma$. Set

$$B(r)=\{u\in L^2(\Omega);\ \|u\|_2\leqq r\},\quad \mathring{B}_\infty(r)=\{u\in L^2(\Omega);\ \|u\|_\infty<r\} \tag{3.1}$$

where $\|u\|_2$, $\|u\|_\infty$ denote the norm in $L^2(\Omega)$, $L_\infty(\Omega)$, respectively. Let us consider the problem

$$u_t(t,x)=\Delta_x u(t,x)+f(t,u(t,x)),\qquad \text{a.e. on } (0,T)\times\Omega, \tag{3.2}$$

$$u(t_0,x)=u_0(x),\qquad \text{a.e. on } \Omega, \tag{3.3}$$

$$u(t,x)=0,\qquad \text{a.e. on } (0,T)\times\Gamma,\quad t^{1/2}u_t\in L^2(0,T;L^2(\Omega)), \tag{3.4}$$

where $t_0\in[0,1)$, $T\in(0,1]$ and $u_0\in L_\infty(\Omega)$. The function $f:[0,1]\times R\to R$ is continuous and $\Delta$ is the Laplace operator in $R^m$. We shall reduce the above problem to the abstract Cauchy problem

$$u'=Au+F(t,u),\quad u(0)=u_0\in B_\infty(1),\qquad t\in[0,1] \tag{3.5}$$

with $A = \Delta$ and $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and $F$ — the Nemytski operator

$$(3.6)\qquad (F(t,u))(x) = f(t,u(x)), \qquad \text{a.e. on } \Omega, \quad t \in [0,1], \quad u \in L^2(\Omega).$$

In [1] and [12] one assumes that for each $t > 0$, the operator $u \to F(t,u)$ is defined on open subsets of $L^2(\Omega)$ (this fact holds if $f$ satisfies a growth condition with respect to $x$). In this paper we do not assume a growth condition of $f$ in $x$, and therefore $F(t,\cdot)$ may fail to be defined on open subsets of $L^2(\Omega)$. Clearly, $F(t,\cdot)$ is defined on $\mathring{B}_\infty(r)$, which is not open in $L^2(\Omega)$. Even more, $\mathring{B}_\infty(r)$ in not locally closed in $L^2(\Omega)$. Of course $F(t,\cdot)$ is defined on $B_\infty(r)$, which is closed in $L^2(\Omega)$, but for $D = B_\infty(r)$ the subtangential condition (A5)″ in Corollary 2.1 is not verified in general. It is interesting that we are not obliged to work in $X = L_\infty(\Omega)$. By applying Theorem 1.1 we can solve (3.5) directly in $X = L^2(\Omega)$. Indeed, if we take $D(t) = B_\infty(t)$ then (A2) holds and $F(t,\cdot)$ is defined on $D(t)$. Suppose for simplicity that

$$(3.7)\qquad |f(t,s)| \leqq 1, \qquad t \in [0,1], \quad s \in [-1,1].$$

This implies $\|F(t,u)\|_\infty \leqq 1$, $\forall t \in [0,1]$, $u \in B_\infty(1)$. Consequently

$$\|S(h)u + hF(t,u)\|_\infty \leqq \|u\|_\infty + h \leqq t + h, \qquad \forall u \in D(t),$$

i.e.

$$S(h)u + hF(t,u) \in D(t+h), \qquad \forall u \in D(t), \quad h > 0, \quad t \in h < 1, \quad t \in [0,1).$$

Thus we have $d(S(h)u + hF(t,u); D(t+h)) = 0$ in $L_\infty(\Omega)$ (hence in $L^2(\Omega)$ too) so (A5) holds in $X = L^2(\Omega)$. Finally, suppose also that for each fixed $t \in [0,1]$, the function $s \to f(t,s)$ is decreasing on $[0,1]$. This implies that $F$ satisfies (A4) in $L^2(\Omega)$ with $g = 0$. Therefore all the hypotheses of Theorem 1.1 are fulfilled, so (3.5) has a unique mild solution $u$ defined on $[0,T]$ for some $T \in (0,1]$, with $u(t) \in D(t)$ on $[0,T]$. Since in this case $A = \Delta$ is the subdifferential of a lower semicontinuous convex function, the mild solution $u$ is even a strong solution in $L^2(\Omega)$ (see e.g. [15], ch. 3) and in addition:

$$u(t) \in D(A) \qquad \text{a.e. on } [0,T], \quad \text{and} \quad t^{1/2}u_t \in L^2(0,T; L^2(\Omega)).$$

In other words we have proved the following result:

THEOREM 3.1. *Let $f: [0,1] \times [-1,1] \to R$ be a continuous function, which satisfies* (3.7) *and is decreasing with respect to the second variable. Then for each $u_0 \in B_\infty(t_0)$ the problem* (3.2)–(3.4) *has a unique solution.*

REMARK 3.1. The new fact here is that $f(t,x)$ does not satisfy a growth condition with respect to the second variable $x$, and therefore the Nemytski operator (3.6) is not defined on open sets of $L^2(\Omega)$. It is defined on closed sets of $L^2(\Omega)$ and (3.5) is regarded as a problem with $t$-dependent domain.

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